Alexander W. N. Riasanovsky **Research Statement**

My main research interests revolve around extremal graph theory as well as the intersection of graph theory and combinatorics with other areas of mathematics such as analysis, optimization, linear algebra, and probability theory.

Much of my recent work revolves around the *edit distance problem* which investigates the furthest a graph can be from satisfying a graph property. There is a great deal of overlap with graph limits in this project as well as spectral graph theory. In another ongoing project, my collaborators and I have made progress on the spread conjecture, a 20-year-old conjecture in extremal spectral graph theory. Other recent projects involve computational and theoretical aspects of spectral graph theory, as well as analysis, optimization, and probability theory.

1. The edit distance problem

Background on the edit distance problem. Given a fixed graph F, how much "work" is needed to remove all induced F from some large host graph G? Here, we measure work with the *edit distance* $\mathrm{dist}(G, H)$, a distance metric on the space of graphs, which is found by overlaying the vertices of graphs G and H, and counting the edges present in one graph, but not the other. To make this problem more precise, we consider $\mathcal{H} = \mathrm{Forb}(F)$ which is the *principal hereditary property* defined to be the class of all graphs which contain no induced copy of F. What is $\max \mathrm{dist}(G, \mathcal{H})$, taken over all graphs on n vertices? While the case where F is a complete graph is neatly answered by the classic result Turán's Theorem, many complicated structures emerge in the general case (see Figure 1). These complications arise from applications of Szeméredi's Regularity Lemma, a central tool in extremal combinatorics, and also require the use of quadratic programming for explicit calculation.

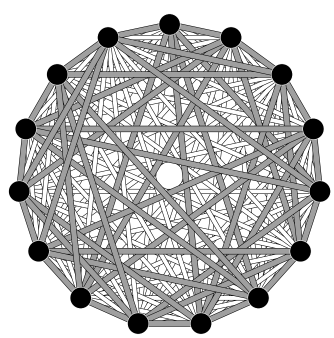

Figure 1. A "complication" which plays a crucial role in understanding the edit distance problem in the case where $F = K_{2,4}$ is forbidden.

The edit distance function and its computation. How big can $\mathrm{dist}(G, \mathcal{H})$ be, if G has edge density p? By including this variation into the edit distance problem, we Extending work of Alon and Stav [1], Balogh and Martin [3] showed that $\mathrm{ed}_{\mathcal{H}}$ can be computed in terms of *colored regularity graphs (CRGs)*, which are structures derived from applications of Szemerédi's Regularity Lemma. A CRG is a complete graph whose vertices are colored either black or white, and whose edges are either black, white, or gray. For any hereditary property $\mathcal{H}$, there is a set $\mathcal{K}_{\mathcal{H}}$ of CRGs which are naturally associated to $\mathcal{H}$.

A natural "editing procedure" leads to the following definitions. Given any CRG K, its *weighted adjacency matrix*, $M_K(p)$ has at entry uv the value $p, 1-p$, or 0, depending on the color of uv as white, black, or gray, respectively. Furthermore for any vector $\mathbf{x} = (x_1, \ldots, x_{|V(K)|})$ of nonnegative "weights" summing to 1, let $g_K(p, \mathbf{x}) := \mathbf{x}^T M_K(p)\mathbf{x}$, and also define $g_K(p) := \min_{\mathbf{x}} g_K(p)$, taken over all such weight vectors. Combined results of Marchant and Thomason [10] and Balogh and Martin [3] reduces the problem entirely to this CRG calculation: for all $p \in [0,1]$, $\mathrm{ed}_{\mathcal{H}}(p) = \min_{K\in\mathcal{K}_{\mathcal{H}}} g_K(p)$.

The edit distance function of a random graph. In recent work, Martin and I found the edit distance function for the hereditary property $\mathcal{H} = \mathrm{Forb}(F)$, where F is an Erdős-Rényi random graph of density p_0, for any $p_0 \in [1-1/\phi, 1/\phi]$. Here ϕ denotes the Golden Ratio. Our result gave an asymptotic formula for for $\mathrm{ed}_{\mathcal{H}}(p)$ which holds with probability tending to 1 and confirms a conjecture from [11]. My main contribution to the proof was two-fold: first, I established that spectral data of the "underlying graph" of a any CRG K can be used to remove K from the minimum expression in $\mathrm{ed}_{\mathcal{H}}(p) = \min_{K\in\mathcal{K}_{\mathcal{H}}} g_K(p)$. Second, I related $\mathrm{ed}_{\mathcal{H}}$ to a result of Bollobás and Thomason [5] which establishes the "$\mathcal{H}$-chromatic number" of a random graph of density p_0. The first development on CRGs is discussed below and has resolved other problems related to the edit distance problem.

New techniques for computing edit distance function. A CRG K is said to be *p-core* if $g_K(p) = \min_{\mathbf{x}} \mathbf{x}^T M_K(p)\mathbf{x}$ defined above is attained by a vector with full support. These CRGs necessarily meet the minimum in the equation $\mathrm{ed}_{\mathcal{H}}(p) = \min_{K\in\mathcal{K}_{\mathcal{H}}} g_K(p)$. Previous work of Marchant and Thomason [10] classifies all p-core CRGs on 2 vertices as well as all 1/2-core CRGs. A naive application of the spectral CRGs result I found classifies all p-core CRGs on an interval of positive length centered 1/2, confirming a conjecture of Martin in [11]. With Martin [12], we extended this classification interval to $[1-1/\phi, 1/\phi]$, where ϕ is the Golden Ratio.

2. The spread conjecture for graphs

Progress on the spread conjecture, using graph limits. In a recent collaboration, my colleagues and I have made progress on the Spread Conjecture for graphs, a problem discussed in more than 50 papers (see [2]). In spectral graph theory, we associate graphs with matrices such as the adjacency matrix and infer properties of the graphs from the eigenvalues of its associated matrix. The *Spread Conjecture*, a problem from extremal spectral graph theory, was first investigated by Gregory, Hershkowitz, and Kirkland [7]. It asks for the graph G which maximizes the difference between the maximum and minimum adjacency eigenvalues attained by a graph.

My main contribution to the project was to apply graph limits, the analytic theory of large graphs, to reduce the Spread Conjecture to a 6-dimensional optimization problem. We aim to complete the proof over the course of the next few months using techniques from numerical analysis.

3. Other recent work

Moments of weighted Cantor measures. In [8], my colleague Steven Harding and I investigated moments of so-called *weighted Cantor measures*. These moments play a central role in understanding the associated L^2-spaces which lack a nice exponential basis. By providing formulas and algorithms both to compute and approximate these moments, we establish techniques for finding Legendre polynomials as an alternative to exponential bases.

We begin with a nonnegative vector $\alpha = (\alpha_1, \ldots, \alpha_N)$ of "weights" summing to 1 and recursively distribute the probability mass on the intervals $[kN^{-\ell}, (k+1)N^{-\ell}] \subseteq [0,1]$ along its equitable

N-interval partition, in proportion to α. The ternary Cantor measure is the case where $\alpha = (1/2, 0, 1/2)$. In general, the moments of a weighted Cantor measure are given by the simple expression

$$\int_{x\in[0,1]} x^k \, d\mu_\alpha(x).$$

Hadamard diagonalizable graphs of order at most 36. A *Hadamard matrix* H is an $n \times n$ matrix with entries either ± 1 such that $HH^T = nI_n$. Hadamard matrices show up in numerous areas of mathematics such as algebra, combinatorics, design theory, and error-correcting codes. They may only exist only for orders $1, 2$, and multiples of 4, (conjectured to exist for all order a multiple of 4), and exhaustive lists only exist up to order 32.

In connection to perfect quantum state transfer, Johnston, Kirkland, Plosker, Storey, and Zhang [9] said that a graph G is *Hadamard-diagonalizable (HD)* if its Laplacian matrix is diagonalizable by some Hadamard matrix H. In connection to [4], Barik, Fallat, and Kirkland found all HD graphs of order at most 12. Beginning at the 2019 Graduate Research Workshop in Combinatorics at University of Kansas, my collaborators and I found all Hadamard matrices of order at most 36. My main contribution to the project was theoretical. Exploiting some properties about spectral decompositions and extremal combinatorics, I found the only possible HD graphs of order $n = 8k+4$.

4. Future work

Edit distance functions of inhomogeneous random graphs. I would like to extend the main result from [12] to include some inhomogeneous random graphs of the form $\mathbb{G}(n, W)$ (here W is a graphon), not just graphs generated by the Erdős-Rényi random graph model $\mathbb{G}(n_0, p_0)$. Ats a preliminary step, I have shown that the $\mathcal{H}$-clique number (the order of the largest induced subgraph satisfying $\mathcal{H}$) of $\mathbb{G}(n, W)$ is concentrated around its mean. This extends work of Doležal, Hladký, and Mathé [6] on clique numbers of inhomogeneous random graphs, but it remains to be shown that $\mathcal{H}$-chromatic numbers are equally concentrated.

Edit distance and p-core CRGs Jointly with Ryan Martin (see [12]), we have already classified all p-core CRGs on the interval $[1-\phi^{-1}, \phi^{-1}]$, where ϕ is the Golden Ratio. This interval has length $\approx 0.236...$ and is the first such classification for an interval of positive length. In Fall 2020, as part of the Iowa State University NSF Research Training Group (NSF DMS award #1839918), we are, alongside new trainees and postdocs, extending these results and developing a variety of new techniques for the edit distance problem, with a paper in preparation.

Optimal spread of a graph. My collaborators and I have already shown, using graph limits, that answering this question asymptotically reduces to solving an optimization problem with finitely many variables. Over the next few months, my collaborators and I expect to complete this calculation, appealing to techniques from optimization and numerical analysis.

I have a broad interest in many aspects of combinatorics, specifically at the intersection of combinatorics with analysis. I have been fortunate enough to have training and experience in a variety of different models and methods. Having found many different questions to my taste, I hope to continue broadening my research portfolio in problems revolving around extremal graph theory, spectral graph theory, graph limits, and probability theory.

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